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LETTER TO THE EDITOR

Aperiodic Ising quantum chain and its relation with a classical Ising chain in an inhomogeneous field

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Abstract. A formal correspondence beetwen the surface magnetization of an Ising quantum chain, perturbed by the paper-folding aperiodic sequence, and the partition function of a classical Ising chain in an inhomogeneous external field is derived. The perturbation is marginal and the critical exponent β_s associated with the surface magnetization is a continuous function of the perturbation amplitude. We obtain this exponent by analysing the classical chain.

The discovery of quasi-crystals [1] and the possibility of building artificial layered structures has initiated the theoretical study of aperiodic systems [2]. The influence of such a layered aperiodicity on critical behaviour was clarified by Luck's relevance criterion [3, 4]. According to this criterion, an aperiodic modulation can be irrelevant, marginal or relevant depending on the sign of a crossover exponent involving the correlation length exponent ν and the wandering exponent ω, characteristic of the sequence. The exact results obtained for the surface magnetization of the two-dimensional layered Ising model with several types of aperiodic modulations (irrelevant, marginal and relevant) are in agreement with the Luck criterion [5].

Recently, an unnoticed connection between aperiodic layered quantum Ising chains and directed random walks was pointed out [6]. Here, a new unexpected relationship between the surface magnetic critical behaviour of such an aperiodic quantum system and the free energy density associated with a one-dimensional classical system is presented. The aperiodic sequence under consideration is the so-called paper-folding sequence obtained from substitution rules [7]. Berche *et al* [8] have treated this problem in the extreme anisotropic limit which leads to the consideration of the behaviour of a quantum Ising chain in a transverse field with Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\sum_{k=1}^{\infty}(\sigma_k^z + \lambda_k \sigma_k^x \sigma_{k+1}^x) \tag{1}$$

where σ are Pauli matrices. The aperiodicity is generated through a modulation of the couplings, parametrized by $\lambda_k = \lambda r^{f_k}$, where r is the modulation ratio and $f_k = 0$ or 1 following the sequence. According to Luck's criterion [3–5] the perturbation is marginal [8]. Thus one expects the critical exponents to vary continuously with r.

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In the ordered phase, the surface magnetization m_{s} (associated with the first site on the quantum chain) in a semi-infinite system is given by [9]

$$m_{\mathrm{s}} = \left(1 + \sum_{k=1}^{\infty} \prod_{j=1}^{k} \lambda_j^{-2}\right)^{-1/2}. \tag{2}$$

For the aperiodic system, with $\lambda_k = \lambda r^{f_k}$, this leads to

$$m_{\mathrm{s}} = [S(\lambda, r)]^{-1/2} \qquad S(\lambda, r) = 1 + \sum_{k=1}^{\infty} \lambda^{-2k} r^{-2n_k} \tag{3}$$

where $n_k = \sum_{j=1}^{k} f_j$. The critical coupling λ_{c} follows from [10]

$$\lim_{L\to\infty} \prod_{k=1}^{L} (\lambda_k)_{\mathrm{c}}^{1/L} = 1 \tag{4}$$

leading to $\lambda_{\mathrm{c}} = r^{-1/2}$. It is shown in [8] that the function $S(\lambda, r)$ satisfies the matrix recursion

$$\begin{pmatrix} S_{\mathrm{odd}}(\lambda, r) \\ S_{\mathrm{even}}(\lambda, r) \end{pmatrix} = \begin{pmatrix} \lambda^{-2} r^{-1} & \lambda^{-2} r^{-2} \\ r^{-1} & 1 \end{pmatrix} \begin{pmatrix} S_{\mathrm{odd}}(\lambda^2 r^{1/2}, r) \\ S_{\mathrm{even}}(\lambda^2 r^{1/2}, r) \end{pmatrix} \tag{5}$$

with

$$S(\lambda, r) = S_{\mathrm{odd}}(\lambda, r) + S_{\mathrm{even}}(\lambda, r) \tag{6}$$

where the subscript odd (even) means that the sum defined in equation (3) runs over odd (even) integers only. At this stage, we make the change of variables

$$H = \ln\left(\frac{\lambda_{\mathrm{c}}}{\lambda}\right)^2 \tag{7}$$

which is natural in the problem since it gives the deviation from the critical point. Defining $r = \exp(K)$, relation (5) becomes

$$\begin{pmatrix} S_{\mathrm{odd}}(H) \\ S_{\mathrm{even}}(H) \end{pmatrix} = \begin{pmatrix} \exp(H) & \exp(H - K) \\ \exp(-K) & 1 \end{pmatrix} \begin{pmatrix} S_{\mathrm{odd}}(2H) \\ S_{\mathrm{even}}(2H) \end{pmatrix}. \tag{8}$$

Iterating this equation n times, one obtains

$$\begin{pmatrix} S_{\mathrm{odd}}(H_0) \\ S_{\mathrm{even}}(H_0) \end{pmatrix} = \prod_{i=0}^{n} \mathcal{T}(H_i) \begin{pmatrix} S_{\mathrm{odd}}(H_{n+1}) \\ S_{\mathrm{even}}(H_{n+1}) \end{pmatrix} \tag{9}$$

with $H_0 = H$ and H_i given by $H_i = 2H_{i-1} = 2^i H$. It is clear that $\mathcal{T}(H_i)$ can be seen as the transfer matrix between site i and site $i+1$ of an Ising classical chain in an inhomogeneous field with Hamiltonian

$$-\frac{\mathcal{H}}{k_{\mathrm{B}} T} = \frac{1}{2}\left(\sum_{i=0}^{\infty} H_i (S_i + 1) + K \sum_{i=0}^{\infty} (S_i S_{i+1} - 1)\right) \tag{10}$$

where $S_i = \pm 1$ are classical Ising variables.

We have thus related the surface magnetization (3) to the partition function Z of the classical Ising chain in a field exponentially increasing from the first site towards the bulk.

In the ordered phase, $\lambda > \lambda_{\mathrm{c}}$, $H_0 < 0$ and the classical spins S_i will align very quickly along the field for $i \to \infty$. In particular, for the non-interacting system with $K = 0$ (corresponding to the homogeneous quantum chain since $r = 1$) the profile $\langle S_i \rangle$ on the classical chain exponentially reaches the value -1 from the first site towards the bulk. So the trivial fixed point $\lambda^\star = \infty$ of the renormalization transformation (5) corresponds in the classical system to a completely ordered phase with expectation value $\langle S\rangle = -1$. We argue that this situation remains unchanged for $r \neq 1$ (see equation (A.6) of [8]).

For $\lambda < \lambda_c$, equation (2) is no longer valid. However, one can still examine the classical chain. The situation for $H_0 > 0$ is now the opposite of the previous one and the system will be ordered in the $+1$ direction.

At criticality, the situation is intermediate between the two previous ones and corresponds to an unstable fixed point of the renormalization scheme. From (7) $H = 0$, and all the $\mathcal{T}(H_i)$ reduce to the same $\mathcal{T}$. Then the partition function Z_n of the classical chain with size n is simply given by

$$Z_n = \operatorname{tr} \mathcal{T}^n = \Lambda_1^n \left[1 + \left(\frac{\Lambda_2}{\Lambda_1}\right)^n\right] \tag{11}$$

with Λ_1 (Λ_2) the largest (smallest) eigenvalue of the transfer matrix $\mathcal{T}$. In the thermodynamic limit the free energy per site is $f(K) = -k_B T \ln \Lambda_1$.

The finite-size behaviour of the surface magnetization (3) leads to

$$S(\lambda_c, r)_L \sim L^{2\beta_s}. \tag{12}$$

The size L of the quantum chain and the size n of the classical chain are related through the relation $L = 2^n$. We can then identify $(2^n)^{2\beta_s} \sim \Lambda_1^n$, which leads to

$$\beta_s = \frac{1}{2}\frac{\ln \Lambda_1}{\ln 2} = \frac{1}{2}\frac{\ln(1 + r^{-1})}{\ln 2} \tag{13}$$

in agreement with [8]. So we have derived a simple relation between the critical exponent β_s calculated in the quantum Ising chain with a transverse field and the free energy density of a one-dimensional classical Ising model. One may note that the critical exponent $\beta_s = 1/2$ of the ordinary surface transition associated with the homogeneous system is obtained by setting $r = 1$ in equation (13). This means that the surface magnetic behaviour of the homogeneous quantum chain is related to the free energy density of a set of non-interacting classical spins in an inhomogeneous field (see equation (10)), since $r = 1$ implies $K = 0$.

Finally, one could ask if this kind of relationship still holds for other sequences or whether it is peculiar to the paper-folding case. This point is currently under investigation.

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